CSCI 334:
Principles of Programming Languages

Lecture 8: PL Fundamentals IV

Instructor: Dan Barowy

Williams

Announcements

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No class on Tuesday (reading period)

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Midterm exam on Thursday

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Midterm review Q&A:
Here, Tuesday, normal time
(optional!)

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Will post study guide shortly after class

Another way to find redexes

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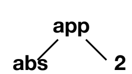

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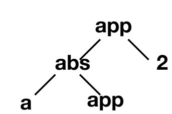

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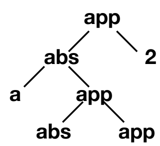

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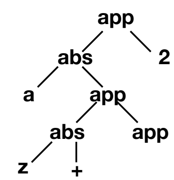

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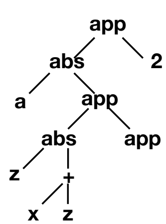

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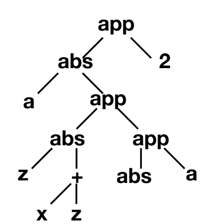

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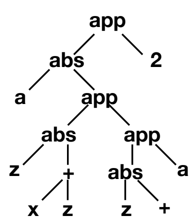

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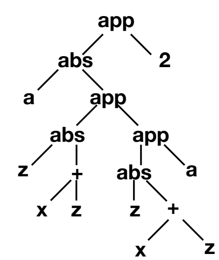

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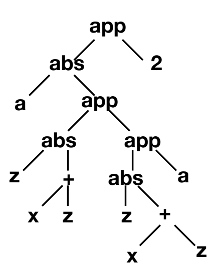

Normal order: “leftmost outermost” application

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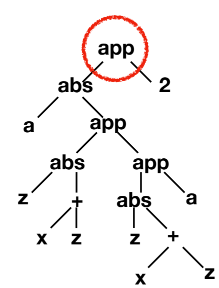

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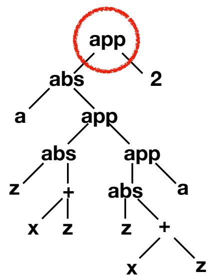

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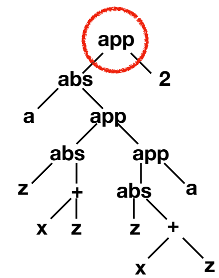

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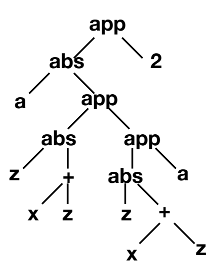

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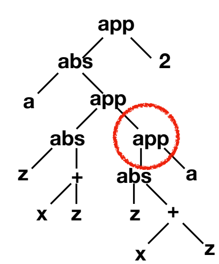

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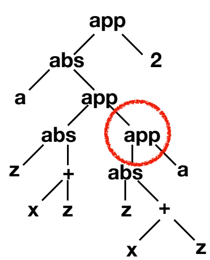

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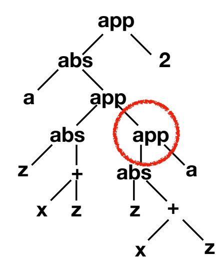

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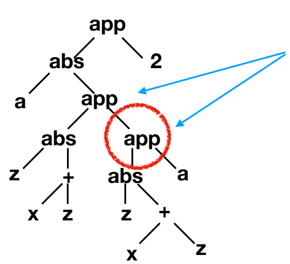

Q: Wait!
Isn’t there a
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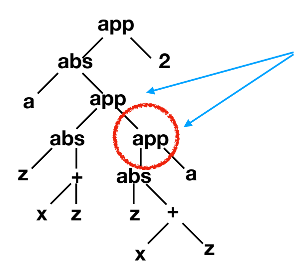

Q: Wait!
Isn’t there a
“more left” app?

A: Yes, but it is not the *innermost*. “Leftmost” is used to break ties among multiple innermost apps.

Applicative order: “leftmost innermost” application

The Halting Problem

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`P(x)` is the output of program `P` run on input `x`.

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How might this work?

Fact: it is provably impossible to write `Halt`

Notes on the proof

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- Reductio ad absurdum (proof form)

Function Evaluation by Substitution

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   return x + 1
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Reductio ad Absurdum

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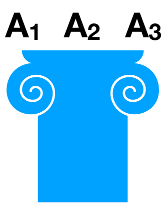

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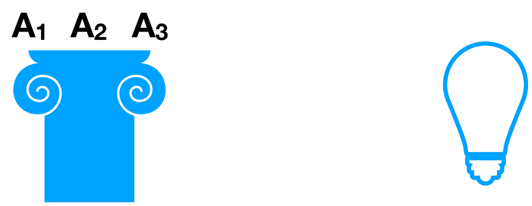
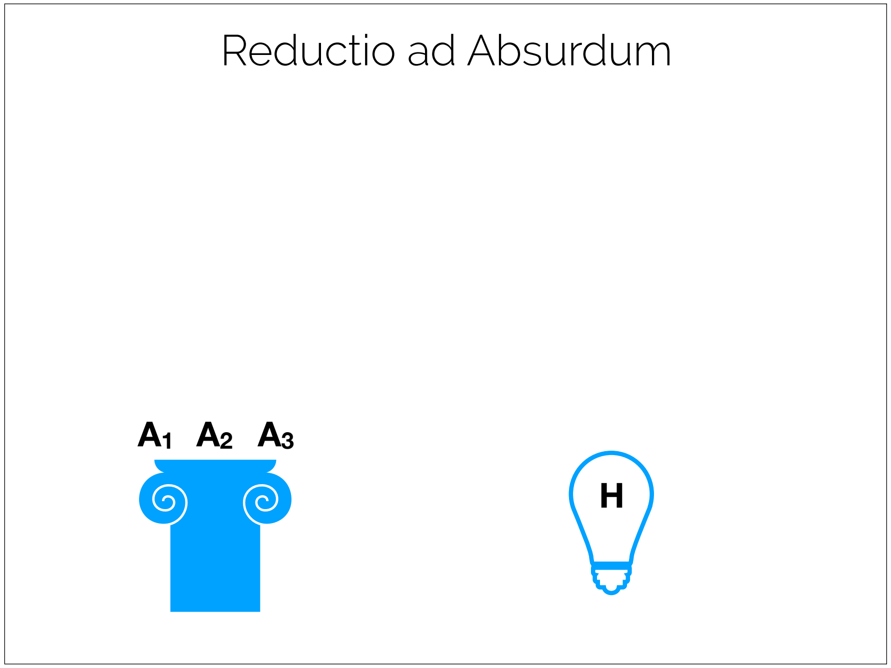
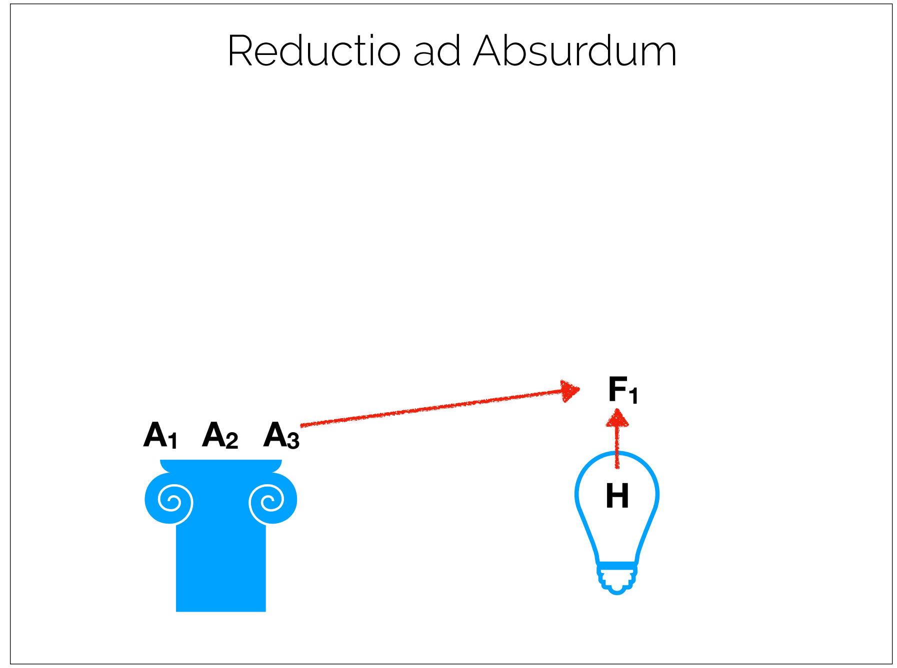



Reductio ad Absurdum
A_1 A_2 A_3
H


Reductio ad Absurdum
F_1
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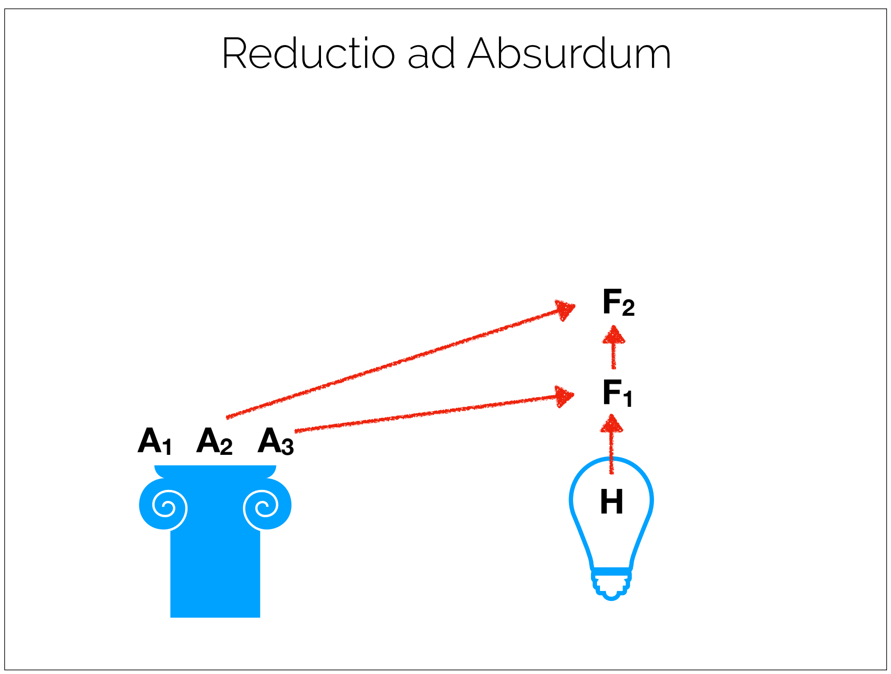
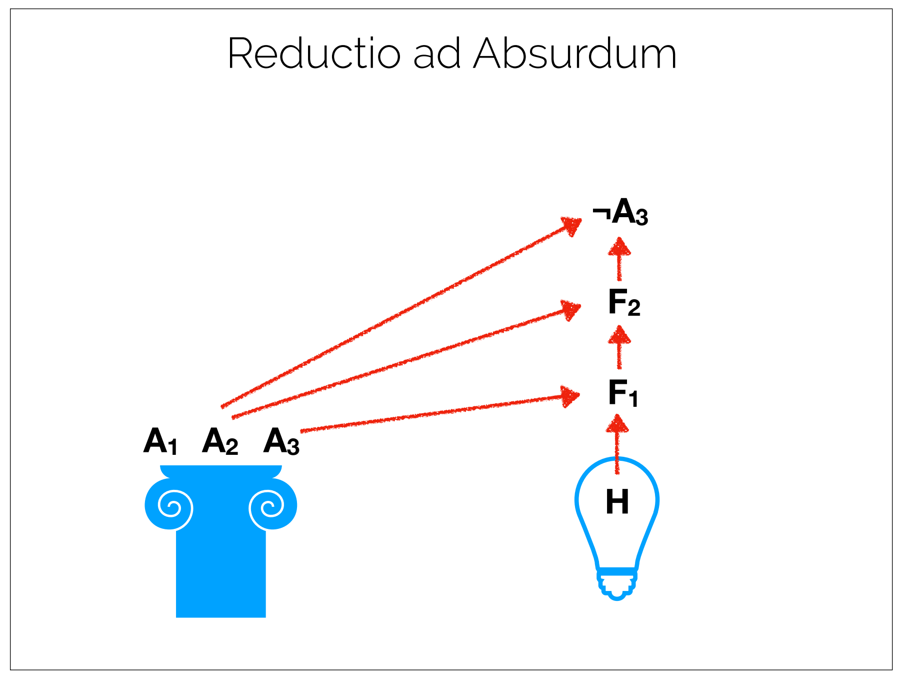
H


Reductio ad Absurdum
F_2
F_1
A_1 A_2 A_3
H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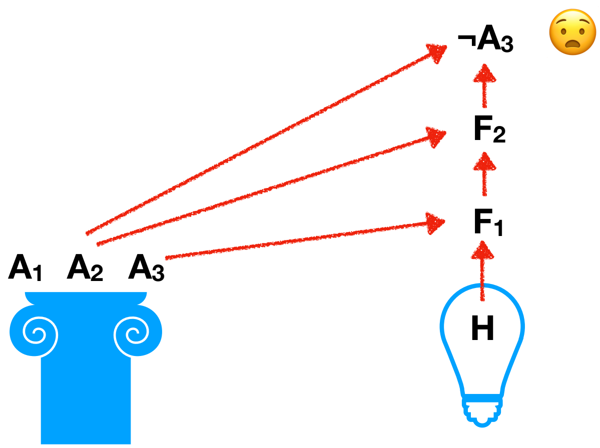

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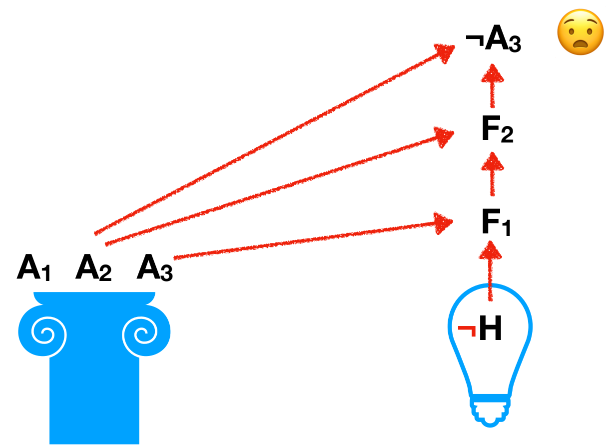

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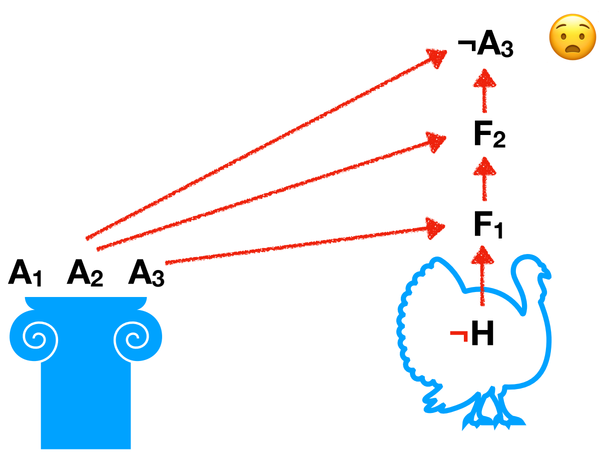

The Halting Problem: Proof

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Suppose:

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Therefore, the `Halt` function cannot exist.

Generality

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```
def myprog(x):
  return 0
```

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def myprog(x):
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```

```
def Halt(P,x):
  if(P = "def myprog(x):\n\treturn 0"):
    return true
  else
    return false
```

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We cannot tell, in general…

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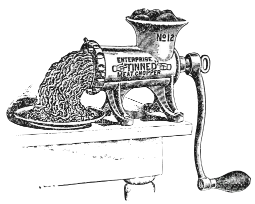
… if a program will eventually produce an error.

… if a program is done using a variable.

Reductions



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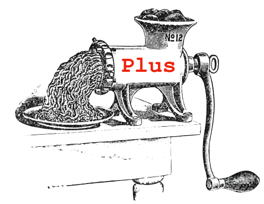

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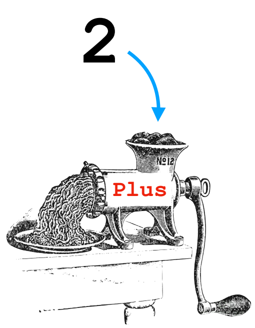

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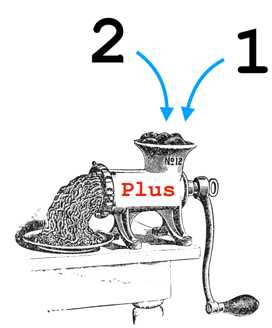

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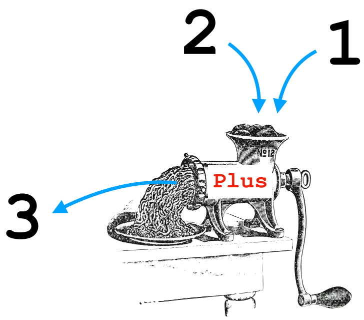

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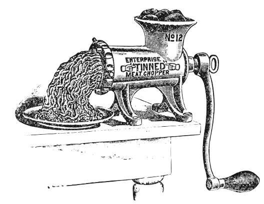

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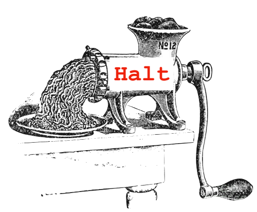

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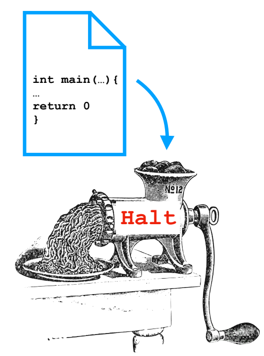

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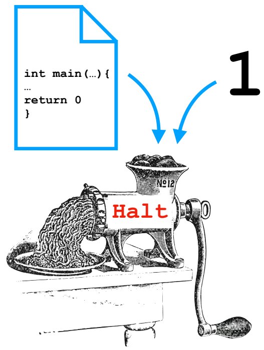

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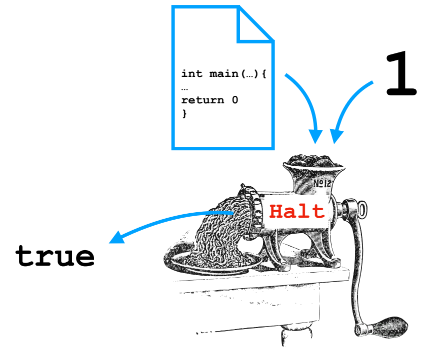

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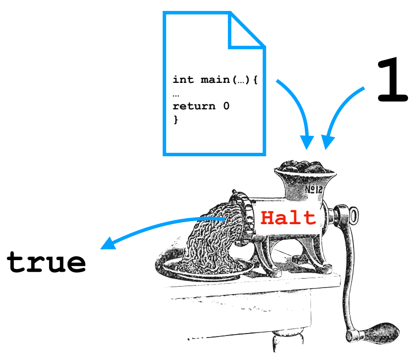
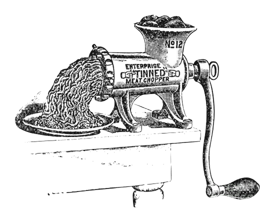


We know that `Halt` is impossible to compute.

Reductions



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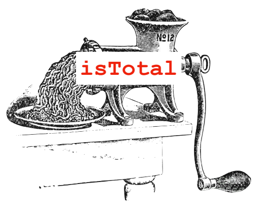

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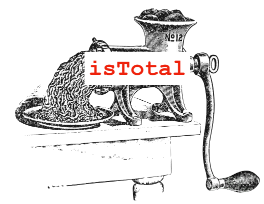
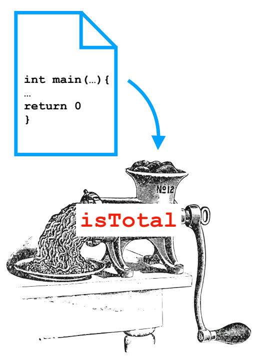


A function `f(i)` is total if and only if `f` is defined for every input `i`.

Reductions



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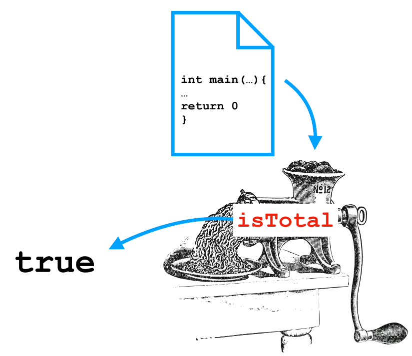

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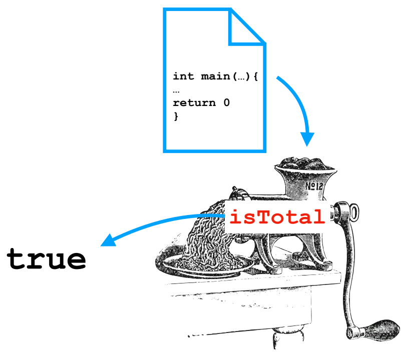


A function `f(i)` is total if and only if `f` is defined for every input `i`.

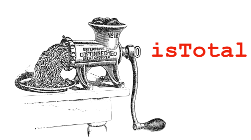
Is `isTotal` computable?

Reductions

Reductions

Assume that `isTotal` is computable.

Reductions



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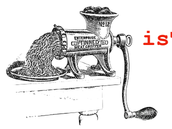
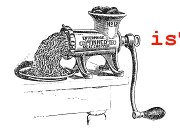


Assume that `isTotal` is computable.

(e.g., it's in your standard library)

Reductions



Assume that `isTotal` is computable.

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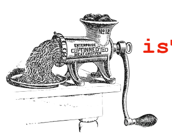
Construct `Halt` using `isTotal`.

Reductions



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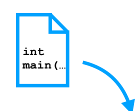

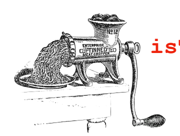

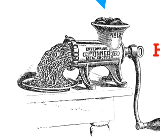

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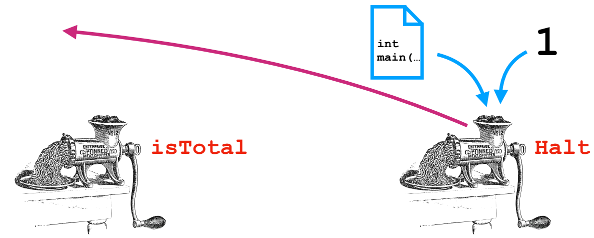

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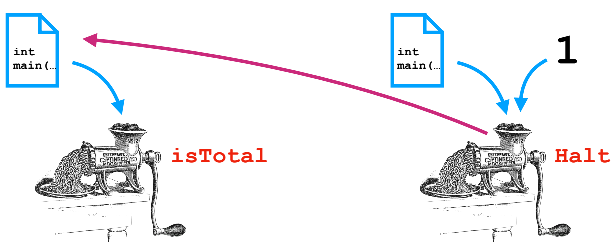

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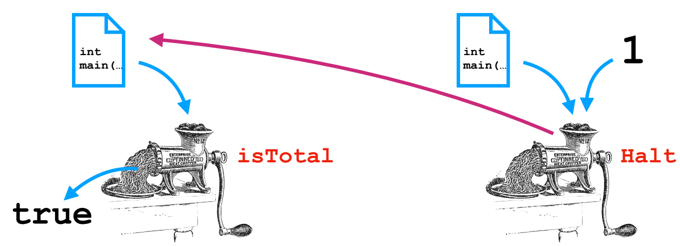

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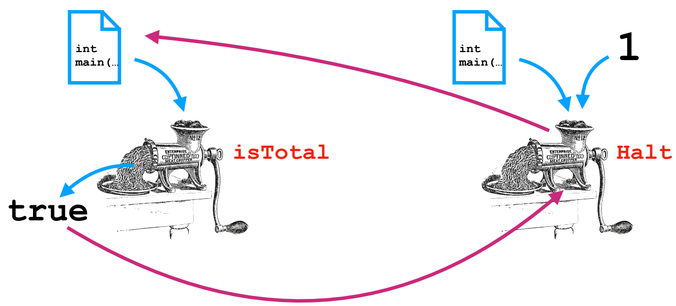

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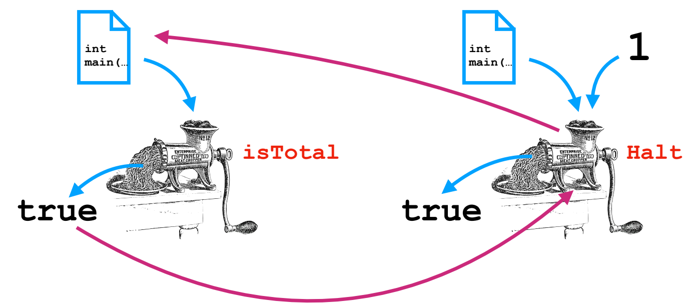

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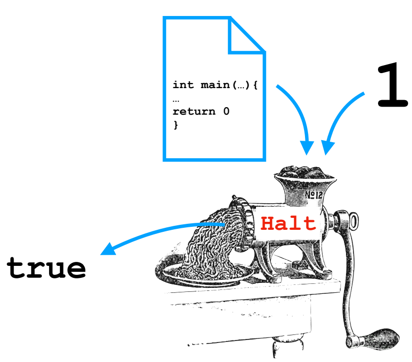

We know that `Halt` is impossible to compute.

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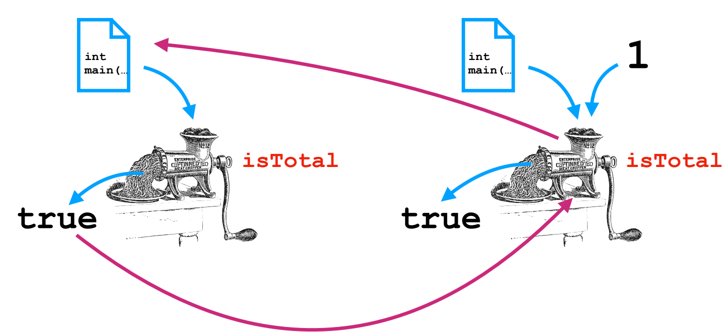
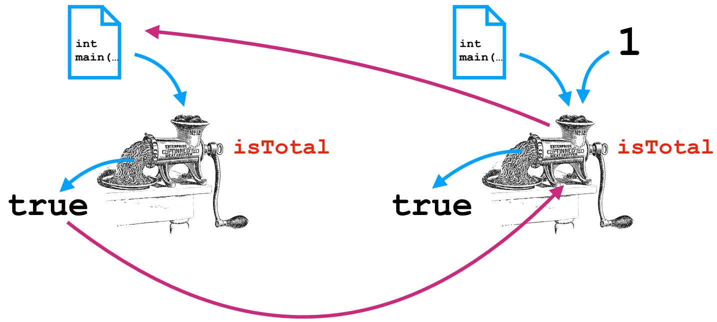


If we can do it, what does this mean for `isTotal`?

Reductions



If we can do it, what does this mean for `isTotal`?

`isTotal` is not computable.

Reductions

Reductions

```
def Halt(p,i):

    return Total(                    )
```

Reductions

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def Halt(p,i):

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Why doesn't this version do what we want?

Reductions

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Because `Total` tells us whether `p` halts on all inputs.

Reductions

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Why doesn't this version do what we want?

Because `Total` tells us whether `p` halts on all inputs.

`Halt` asks specifically about input `i`.

Reductions

```
def Halt(p,i):


    return Total(                  )
```

Reductions

```
def Halt(p,i):
    def Runs(j):
        p(i)
        return "done!"
    return Total(                  )
```

Reductions

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def Halt(p,i):
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        p(i)
        return "done!"
    return Total( getsource(Runs) )
```

Reductions

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This is a valid program.

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So we can solve the Halting Problem!

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    return Total( getsource(Runs) )
```

This is a valid program.

So we can solve the Halting Problem!

Therefore, `Total` is not computable. 😞

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What have we learned?

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- Basic syntax

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- Program evaluating using reduction rules

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Not on the exam:

Mini-Midterm Review

Not on the exam:

- Halting problem
- Reductions